PMM U.S.S.R., Vol.45,pp.707-709 0021-8928/82/5 0707 $7.50/0

UDC 539.4.012

EXTENSION OF AN ELASTIC HALF-SPACE WITH A CRACK PERPENDICULAR TO ITS SURFACE*

B.I. SMETANIN and B.V. SOBOL'

The problem of elastic equilibrium of a half-space weakened by a plane crack is considered. The crack is perpendicular to the surface of the half-space. A normal load symmetrical with respect to the crack plane is applied to the crack edges. The problem is reduced to solving an integral equation of the first kind. For the case of an elliptical crack, a solution of this equation is obtained in a simple form suitable for practical applications. The problem was solved numerically earlier in /1,2/.

1. Let us denote the region occupied by the crack in the plane $y=0$ by Ω. The crack is opened by the action of the load $\sigma_y=-p(x,z)$ $(y=\pm 0;\ x,z\in\Omega)$ (the plus and minus signs refer to the right and left edge of the crack). The boundary $z=0$ of the half-space $z\geqslant 0$ shall be assumed, for definiteness, stress-free. The stresses and displacements vanish when $z\to\infty$. The ideas expounded in /3/ are used to reduce the problem to an integral in terms of the function $\chi(x,z)=V|_{y=+0}-V|_{y=-0}$ characterizing the opening of the crack edges. Here and henceforth U, V and W denote the projections of the displacement vector on the x, y and z axis respectively. Applying the Fourier transformation in x and y to the Lamé equations, we obtain a system of six linear ordinary differential equations which can be written in vector form as follows:

$$d\mathbf{L}/dz+\mathbf{PL}=\mathbf{F}(\Psi) \tag{1.1}$$

The components of the vector function $\mathbf{L}$ are Fourier transforms in the variables x, y of the functions U, V and W, and their first derivatives in z. The elements of the matrix $\mathbf{P}$ are given by the parameters α and β of the Fourier transformation. The function Ψ has the form

$$\Psi(\alpha,z)=\int_{-\infty}^{\infty}\chi(x,z)e^{i\alpha x}dx$$

The equation (1.1) should be solved with the conditions of the absence of stress at $z=0$ taken into account.

The method of constructing the solution of (1.1) is given in detail in /3/. Realization of this scheme enables us to obtain, after certain manipulations, the following representation for the function $S(\alpha,\beta,z)$ (the derivation is bulky and is therefore omitted)

$$S(\alpha,\beta,z)=\mu\int_0^{\infty}\Psi(\alpha,\zeta)\left[K_1(\alpha,\beta,z-\zeta)+K_2(\alpha,\beta,z,\zeta)\right]d\zeta \tag{1.2}$$

$$S(\alpha,\beta,z)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sigma_y(x,y,z)e^{i(\alpha x+\beta y)}dxdy$$

$$K_1(\alpha,\beta,\eta)=\frac{\lambda+\mu}{\lambda+2\mu}\left[\frac{\beta^4\gamma|\eta|+\beta^4-4\gamma^2\beta^2}{\gamma^3}+4\delta(\eta)\right]e^{-\gamma|\eta|}$$

$$K_2(\alpha,\beta,z,\zeta)=\frac{1}{\lambda+2\mu}\left[-\frac{2\lambda\gamma^2+(\lambda+3\mu)\beta^2}{\gamma^2}\beta(z+\zeta)+\right.$$
$$\left.\frac{2(\lambda+\mu)\beta^4}{\gamma}z\zeta+\frac{2\lambda^2\gamma^4+4(\lambda^2+3\lambda\mu+\mu^2)\gamma^2\beta^2-(\lambda^2+2\lambda\mu-\mu^2)\beta^4}{(\lambda+\mu)\gamma^3}\right]\times e^{-\gamma(z+\zeta)}$$

where $\gamma=\sqrt{\alpha^2+\beta^2}$, λ,μ are the Lamé constants and $\delta(\eta)$ is the Dirac delta function.

Applying to (1.2) the convolution theorem and returning to the original functions, we

*Prikl.Matem.Mekhan.,45,No.5,940-943,1981 obtain the following integral equation for the function χ:

$$\Delta \iint_{\Omega} \chi(\xi, \zeta) k_1(x-\xi, z-\zeta)\, d\Omega + \iint_{\Omega} \chi(\xi, \zeta) k_2(x-\xi, z, \zeta)\, d\Omega = -\pi \frac{p(x, z)}{\mu} \quad (x, z \in \Omega) \tag{1.3}$$

where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}, \quad k_1(\eta, t) = \frac{\lambda+\mu}{2(\lambda+2\mu)} (\eta^2 + t^2)^{-1/2} \tag{1.4}$$

$$k_2(\eta, z, \zeta) = \frac{9(\lambda+\mu)}{\lambda+2\mu} \frac{z\zeta}{[(z+\zeta)^2+\eta^2]^{5/2}} -$$

$$\frac{6\mu}{\lambda+\mu}(z+\zeta) \frac{[(z+\zeta)^2+\eta^2]^{5/2} - (z+\zeta)^5 - {}^5/_2 (z+\zeta)^3 \eta^2}{\eta^4 [(z+\zeta)^2+\eta^2]^{5/2}} -$$

$$\frac{\lambda-7\mu}{2(\lambda+2\mu)} \frac{\eta^2}{[(z+\zeta)^2+\eta^2]^{5/2}} - \frac{\lambda^2-18\lambda\mu-43\mu^2}{2(\lambda+\mu)(\lambda+2\mu)} \frac{(z+\zeta)^2}{[(z+\zeta)^2+\eta^2]^{5/2}}$$

We note that the kernel k_1 corresponds to the problem of a crack in a space, and the singularity in the kernel k_2 is removable. If Ω is a strip defined by the conditions $0 < z_0 \leqslant z \leqslant z_1$, $|x| < \infty$ and p is independent of x, then (1.3) is transformed into the integral equation of the corresponding plane problem /4/.

2. We can use the asymptotic method /5/ to solve equation (1.3). We shall assume for definiteness that Ω is an ellipse with the axes equal to $2a$ and $2b$. The axis $2b$ is parallel to the plane $z = 0$ and lies at the distance h from it. We place the coordinate origin at the ellipse center. Without introducing new notation for χ and p, we obtain from (1.3)

$$\Delta \iint_{\Omega} \chi(\xi, \zeta) k_1(x-\xi, z-\zeta)\, d\Omega + \iint_{\Omega} \chi(\xi, \zeta) k_2(x-\xi, z+h, \zeta+h)\, d\Omega = -\pi \frac{p(x, z)}{\mu} \quad (x, z \in \Omega) \tag{2.1}$$

The kernels k_1 and k_2 of (2.1) have, as before, the form (1.4). Let us introduce the parameters $\varepsilon = a/h$ $(0 \leqslant \varepsilon < 1)$ and write the kernel k_2 in the form of the following expansion:

$$k_2(\eta, z+h, \zeta+h) = \sum_{n=3}^{\infty} k_{2n}(\eta, z, \zeta)\, \varepsilon^n \tag{2.2}$$

We shall seek the solution of (2.1) in the form of the following asymptotic series in powers of ε /5/:

$$\chi(x, z) = \sum_{m=0}^{\infty} \chi_m(x, z)\, \varepsilon^m \tag{2.3}$$

Substituting (2.2) and (2.3) into (2.1) and equating the expressions accompanying like powers of ε, we arrive at the following infinite system of integral equations:

$$\begin{aligned} &\Delta \iint_{\Omega} \chi_0(\xi, \zeta) k_1(x-\xi, z-\zeta)\, d\Omega = -\pi \frac{p(x, z)}{\mu} \\ &\Delta \iint_{\Omega} \chi_1(\xi, \zeta) k_1(x-\xi, z-\zeta)\, d\Omega = 0 \\ &\Delta \iint_{\Omega} \chi_2(\xi, \zeta) k_1(x-\xi, z-\zeta)\, d\Omega = 0 \qquad\qquad (x, z) \in \Omega \\ &\Delta \iint_{\Omega} \chi_3(\xi, \zeta) k_1(x-\xi, z-\zeta)\, d\Omega = -\iint_{\Omega} \chi_0(\xi, \zeta) k_{23}(x-\xi, z, \zeta)\, d\Omega \end{aligned} \tag{2.4}$$

etc.

Making use of the results obtained by Galin in /6/, we can show that if

$$p(x, z) = \sum_{i=0}^{r} \sum_{j=0}^{l} p_{ij} x^i z^j \quad \begin{pmatrix} r+l=n \\ p_{ij} \text{ are const} \end{pmatrix} \tag{2.5}$$

then the solution of the first equation of (2.4) has the following form for the elliptical region Ω:

$$\chi_0(x,z)=(1-z^2/a^2-x^2/b^2)^{1/2}\sum_{i=0}^{r}\sum_{j=0}^{l} q_{ij}x^iz^j \quad \begin{pmatrix} r+l=n \\ q_{ij} \text{ are const} \end{pmatrix} \tag{2.6}$$

The coefficients q_{ij} can be expressed in terms of p_{ij} according to a scheme given for instance in /5/.

Let us consider the case $p(x,z)=p=\text{const}$. Solving the system (2.4) and using the formulas of the form (2.5) and (2.6), we obtain

$$\chi(x,z)=\frac{2Ap}{\theta E(k)}(1-z^2/a^2-x^2/b^2)^{1/2}\times\left\{1+B\frac{c\varepsilon^3}{8}\left[\frac{2}{3E(k)}-\frac{\varepsilon k^2 z}{aD(k)}\right]+O(\varepsilon^5)\right\} \tag{2.7}$$

$$A=\begin{cases} a, & a\leqslant b \\ b, & a>b \end{cases},\quad B=\begin{cases} b/a, & a\leqslant b \\ b^2/a^2, & a>b \end{cases}$$

$$D(k)=\begin{cases} (k^2+1)E(k)+(k^2-1)K(k), & a\leqslant b \\ (2k^2-1)E(k)+(k^2-1)K(k), & a>b \end{cases}$$

$$k=\begin{cases} (1-a^2/b^2)^{1/2}, & a\leqslant b \\ (1-b^2/a^2)^{1/2}, & a>b \end{cases},\quad c=\frac{7\lambda^2+9\lambda\mu+5\mu^2}{4(\lambda+\mu)^2}$$

Here $K(k)$, $E(k)$ are the complete elliptic integrals of the first and second kind respectively, $\theta=\mu(1-\nu)^{-1}$ and ν is the Poisson's ratio. From (2.7) we obtain

$$N=1+B\frac{c\varepsilon^3}{8}\left[\frac{2}{3E(k)}-\frac{\varepsilon k^2\cos\varphi}{D(k)}\right]+O(\varepsilon^5) \tag{2.8}$$

where $N=K_I/K_{I0}$, K_I is the normal stress intensity coefficient for the case in question, and K_{I0} for the case of a crack in a space $(\varepsilon=0)$ /7/. The angle φ is counted from the positive direction of the z-axis.

Table 1

a/b	$\varphi=\pi$		$\varphi=0$	
	(2.8)	/2/	(2.8)	/2/
1	1.01	1.015	1.00	1.005
0.5	1.03	1.033	1.01	1.013
0.25	1.07	1.055	1.03	1.026

Calculations show that the formulas (2.7) and (2.8) can be used, with sufficient practical accuracy, for $0\leqslant\varepsilon\leqslant 0.5, a/b\geqslant 0.25$ and $0\leqslant\nu\leqslant 0.5$. The Table 1 gives the values of N calculated using (2.8) and obtained in /2/ for $\varepsilon=0.5, \nu=0$.

REFERENCES

1. SHAH R.C. and KOBAYASHI A.S., Stress intensity factors for an elliptical crack approaching the surface of a semi-infinite solid. - Internat. J. Fract., Vol.9, No.2, 1973.
2. NISITANI A. and MURAKAMI Y., Stress intensity factors of an elliptical crack or a semi-elliptical crack subject to tension. —Internat.J. Fract., Vol.10, No.3, 1974.
3. POPOV G.Ia., On a method of solving mechanics problems for domains with slits or thin inclusions. PMM, Vol.42, No.1, 1978.
4. PANASIUK V.V., SAVRUK M.P. and DATSYSHIN A.P., Stress Distribution Around Cracks in Plates and Shells. Kiev, NAUKOVA DUMKA, 1976.
5. VOROVICH I.I., ALEKSANDROV V.M. and BABESHKO V.A., Nonclassical Mixed Problems of the Theory of Elasticity. Moscow, NAUKA, 1974.
6. GALIN L.A., Contact Problems of the Theory of Elasticity. Moscow, GOSTEKHIZDAT, 1953.
7. CHEREPANOV G.P., Mechanics of Brittle Fracture, Moscow, NAUKA, 1974.

Translated by L.K.